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# Exact solution of the three-color problem on a random lattice

Ivan K. Kostov

*C.E.A.-Saclay, Service de Physique Théorique, F-91191 Gif-Sur-Yvette, France*

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## Abstract

We present the exact solution of Baxter's three-color problem on a random planar graph, using its formulation in terms of three coupled random matrices.

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## 1. Introduction

Baxter's three-color problem on a regular hexagonal lattice [1] is one of the classical examples of solvable lattice models. The free energy of this model is equal to the number of coloring the links of the lattice with three different colors,  $A$ ,  $B$  and  $C$ , so that no links that meet at a vertex carry the same color. Alternatively one can speak of the number of three-coloring of the links of the regular triangulated lattice in a way that the three sides of every triangle have different colors. The problem is equivalent to a special version of the  $O(2)$  model, whose partition function is given by a gas of fully packed loops on the hexagonal lattice, having two different flavors. Baxter also showed that this model solves the problem of coloring the faces of the hexagonal lattice with four different colors, so that the adjacent faces have different colors. More recently yet another geometrical interpretation of this model has been found, namely as the problem of counting the different foldings of a regular triangular lattice [2].

The three-coloring problem can be formulated also for a random 3-coordinated planar graph (or a random triangulation, in the dual language) [3]. Here one has the freedom to chose different ensembles of planar graphs. The generating function for all possible three-colored planar graphs is given by the three-matrix integral [4]

$$\mathcal{Z}_N(\beta) = \int_{N \times N} dA dB dC \exp \left\{ -N \operatorname{Tr} \left( \frac{1}{2} (A^2 + B^2 + C^2) - g[ABC + BAC] \right) \right\} \quad (1.1)$$

where  $A$ ,  $B$  and  $C$  are Hermitian  $N \times N$  matrix variables.

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*E-mail address:* [kostov@spht.saclay cea.fr](mailto:kostov@spht.saclay cea.fr) (I.K. Kostov).

The perturbative expansion of the free energy

$$\mathcal{F}(g) \equiv \frac{1}{N^2} \log \mathcal{Z}(g) = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} \mathcal{N}_h(2m) N^{-2h} g^{2m} \quad (1.2)$$

has the following combinatorial meaning:  $\mathcal{N}_h(2m)$  is the number of the connected three-colored triangulations of a surface of genus  $h$ , containing  $2m$  triangles. (The configurations with nontrivial symmetry group are taken with the corresponding symmetry factors.)

The model has still the interpretation as a fully packed  $O(2)$  loop gas model, with the restriction that only loops of even length are allowed. It can be also interpreted as the four-coloring problem of the faces of a random 3-coordinated graph. The interpretation in terms of foldings does not generalize to this ensemble: it holds only for planar graphs whose faces have even number of sides.

An iterative procedure for evaluating the coefficients in the expansion (1.2) has been proposed by Eynard and Kristjansen [5]. They considered, more generally, the  $O(2n)$  loop gas model with only even loops, which reduces to the three-color problem when  $n = 1$ . This  $O(2n)$  model is actually identical to fully packed  $O(n)$  loop gas model defined on a 4-coordinated planar graph.<sup>1</sup> It is, therefore, expected that the matrix model (1.1) is in the class of universality of the dense  $O(1)$  model, which means that  $\mathcal{N}_h(2m)$  have the same large- $m$  asymptotics as the number of the non-colored planar graphs:

$$\mathcal{N}_h(2m) \sim g_*^{-2m} m^{\frac{3}{2}(h-1)-1}, \quad (1.3)$$

where  $g_*$  is the critical value of the coupling  $g$ . The iterative procedure was carried out up to the order 12, with the result

$$\lim_{N \rightarrow \infty} \mathcal{F}(g) = 2g^2/2 + 14g^4/4 + 138g^6/6 + 1608g^8/8 + 20736g^{10}/10 + 286452g^{12}/12 + \dots \quad (1.4)$$

Soon after the work of Eynard and Kristjansen, the exact solution of the matrix model (1.2) has been found for *purely imaginary* coupling  $g$  [6]. An explicit expression for the free energy of the model was obtained in terms of elliptic functions, and it was checked that the coefficients in the expansion (1.4) are correctly reproduced [7]. However, it was not possible to carry out the analytic continuation of this solution to real values of  $g$ .

In this Letter we present the direct solution of the matrix model (1.2) for real coupling  $g$ , using a method similar to the one applied in [6]. We find the expected critical behavior of a  $c = 0$  matter coupled to 2D gravity and evaluate the critical coupling  $g_*$ . Finally we explain why the cases of real and imaginary coupling  $g$  are not related by analytic continuation.

## 2. Saddle point equations for the matrix integral

After integrating over the  $B$  and  $C$  matrices, the partition function reduces to the following integral over the eigenvalues  $\lambda_1, \dots, \lambda_N$  of the matrix  $A$ :

$$\mathcal{Z}_N(\beta) = \beta^{N^2} \int \prod_{k=1}^N \frac{d\lambda_k}{g^{-2} - 4\lambda_k^2} e^{-\frac{1}{2}N\lambda_k^2} \prod_{i < j} \frac{(\lambda_i - \lambda_j)^2}{g^{-2} - (\lambda_i + \lambda_j)^2}. \quad (2.1)$$

The saddle-point spectral density

$$\rho(\lambda) = \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) \right\rangle \quad (2.2)$$

<sup>1</sup> More precisely, each 4-vertex is visited either by one loop going straight, or by two loops turning at right angle.

is an even function supported by a symmetric interval  $[-\Delta, \Delta]$ , where  $\Delta$  is a function of the coupling  $g$ , such that  $\Delta|_{g=0} = 2$ . At  $g = 0$  it is given by the Wigner's semi-circle law:  $\rho(\lambda)|_{g=0} = \frac{1}{2\pi}\sqrt{4-\lambda^2}$ . It is completely determined by the saddle-point equation

$$\lambda = \oint_{-\Delta}^{\Delta} d\lambda' \rho(\lambda') \left[ \frac{2}{\lambda - \lambda'} - \frac{1}{\lambda + \lambda' + g^{-1}} - \frac{1}{\lambda + \lambda' - g^{-1}} \right] \quad (2.3)$$

and the normalization condition  $\int d\lambda \rho(\lambda) = 1$ . Introducing the analytic function

$$W(z) = \int_{-\Delta}^{\Delta} \frac{d\lambda \rho(\lambda)}{z - \lambda}, \quad (2.4)$$

we write the saddle-point equation as a functional condition for  $W(z)$ :

$$W(\lambda + i0) + W(\lambda - i0) + W(-1/g - \lambda) + W(1/g - \lambda) = \lambda, \quad \lambda \in [-\Delta, \Delta]. \quad (2.5)$$

The density  $\rho(\lambda)$  is an even function of  $\lambda$ , which means that (2.4) is antisymmetric:

$$W(z) = -W(-z). \quad (2.6)$$

The saddle point equation can be therefore rewritten as

$$W(\lambda + i0) + W(\lambda - i0) - W(\lambda + 1/g) - W(\lambda - 1/g) = \lambda, \quad \lambda \in [-\Delta, \Delta]. \quad (2.7)$$

The normalization condition for the density fixes the first coefficient in the expansion of  $W(z)$  at infinity:

$$W(z) = \frac{1}{z} + \frac{\langle \frac{\text{Tr}}{N} A^2 \rangle}{z^3} + \mathcal{O}(z^5). \quad (2.8)$$

As mentioned in [5], knowing the quantity  $\langle \frac{\text{Tr}}{N} A^2 \rangle$  is enough to solve the three-coloring problem, because it is related to the free energy by

$$\left\langle \frac{\text{Tr}}{N} A^2 \right\rangle = 1 + g \frac{d}{dg} \mathcal{F}(\beta). \quad (2.9)$$

### 3. Solution of the saddle point equation

The four-term functional Eq. (2.7) with the boundary condition  $W(z) \sim 1/z$  at infinity is sufficient to determine  $W(z)$ . Introducing the function

$$\zeta(z) = 2g^{-1} [W(z + 1/2g) - W(z - 1/2g)] + z^2, \quad (3.1)$$

we write (2.7) as a two-term difference equation

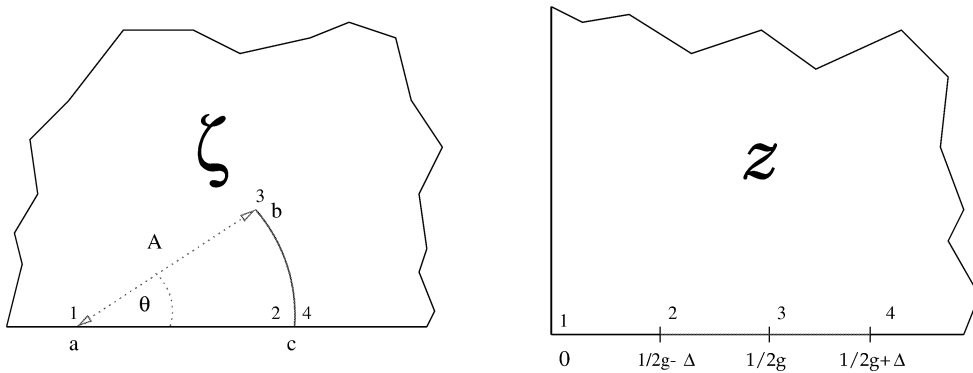
$$\zeta(\lambda + 1/2g \pm i0) = \zeta(\lambda - 1/2g \mp i0), \quad \lambda \in [-\Delta, \Delta]. \quad (3.2)$$

The function  $\zeta(z)$  is real and symmetric:

$$\zeta(z) = \zeta(-z) = \overline{\zeta(\bar{z})}, \quad (3.3)$$

possesses two cuts along the intervals  $[-\Delta - \frac{1}{2g}, \Delta - \frac{1}{2g}]$  and  $[-\Delta + \frac{1}{2g}, \Delta + \frac{1}{2g}]$  on the real axis, and expands at  $z \rightarrow +\infty$  as

$$\zeta(z) = z^2 - \frac{2}{g^2 z^2} - \frac{(\frac{1}{2g^2} + 6W_2)}{g z^4} + \mathcal{O}(z^{-6}). \quad (3.4)$$

Fig. 1. The map  $z \leftrightarrow \zeta$ .

From the symmetry of  $\zeta(z)$  it follows that it is real when  $z$  is real and outside the cuts, and also when  $z$  is purely imaginary. Therefore, the map  $z \rightarrow \zeta$  transforms the quadrant  $\{\text{Im } z > 0, \text{Re } z > 0\}$  into the to the upper semi-plane  $\{\text{Re } \zeta > 0\}$ , cut along an arc connecting the points  $c$  and  $b$ , where  $b$  is some complex number (Fig. 1). At the endpoint  $c$ , the arc should be perpendicular to the real axis. We denote the images of the special points of the map

$$z_1 = 0, \quad z_2 = \frac{1}{2g} - \Delta, \quad z_3 = \frac{1}{2g}, \quad z_4 = \frac{1}{2g} + \Delta, \quad (3.5)$$

by

$$a = \zeta(z_1), \quad b = \zeta(z_3), \quad c = \zeta(z_2) = \zeta(z_4). \quad (3.6)$$

It is easy to see that the inverse function  $z(\zeta)$  can be obtained as the integral<sup>2</sup>

$$z = \frac{1}{2} \int_a^\zeta \frac{dt(t-c)}{\sqrt{(t-a)(t-b)(t-\bar{b})}}. \quad (3.7)$$

The asymptotics (3.4) is satisfied if

$$a + b + \bar{b} - 2c = 0, \quad 2c^2 - a^2 - b^2 - \bar{b}^2 = 6/g^2. \quad (3.8)$$

The integral (3.7) is a standard elliptic integral. Denoting

$$\text{cn } u \equiv \text{cn}(u, k) = \frac{A - \zeta + a}{A + \zeta - a}, \quad b - a = Ae^{i\theta}, \quad k = \cos(\theta/2) \quad (3.9)$$

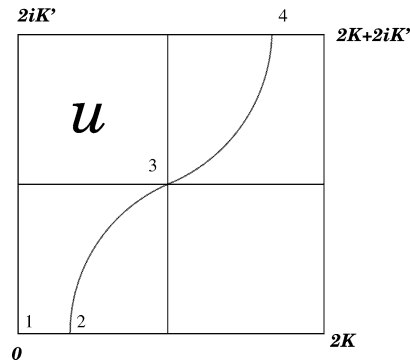
we have ([9], 239.00, 239.07 and 341.53)

$$-\frac{z}{\sqrt{A}} = \frac{a-c-A}{2A}u + \int_0^u \frac{du}{1 + \text{cn } u} = Cu - \frac{H'(u, k)}{H(u, k)} + \frac{\text{dn}(u, k)}{\text{sn}(u, k)} \quad (3.10)$$

where

$$C = \frac{A + a - c}{2A} - \frac{E}{K}. \quad (3.11)$$

<sup>2</sup> Here we are following the argument by Hoppe in [8].

Fig. 2. The domain of the  $u$ -variable.

The elliptic parameters  $u_1, \dots, u_4$  of the special points of the map  $z \rightarrow \zeta$  are (see Fig. 2)

$$u_1 = 0, \quad u_2 \in [0, K], \quad u_3 = K + iK', \quad u_4 = 2u_3 - u_2, \quad (3.12)$$

and the point  $z \rightarrow \infty$  ( $\zeta \rightarrow \infty$ ) corresponds to  $u_\infty = 2iK'$ .

From (3.5) we determine the constants  $A$  and  $C$ :

$$z_3 = z(K + iK') = \sqrt{A} \left[ -CK + i \left( \frac{\pi}{2K} + CK' \right) \right] = \frac{1}{2g}, \quad (3.13)$$

which gives

$$C = -\frac{\pi}{2KK'}, \quad \sqrt{A} = \frac{\beta}{\pi} K'. \quad (3.14)$$

Then, using the Legendre's relation  $EK' + E'K - KK' = \pi/2$ , we write (3.11) as

$$\frac{c-a}{A} = \frac{2E' - K'}{K'}. \quad (3.15)$$

The final formula is

$$\begin{aligned} \tilde{z}(u) &\equiv \frac{z(u)}{\sqrt{A}} = \frac{\pi}{2KK'} u + \frac{H'(u, k)}{H(u, k)} - \frac{\text{dn}(u, k)}{\text{sn}(u, k)}, \\ \tilde{\zeta}(u) &\equiv \frac{\zeta(u) - a}{A} = \frac{1 - \text{cn}(u, k)}{1 + \text{cn}(u, k)}. \end{aligned} \quad (3.16)$$

Note the very useful relation

$$2 \frac{d\tilde{z}}{du} = \tilde{\zeta}(u) - \frac{2E' - K'}{K'}. \quad (3.17)$$

Now it remains to determine the elliptic modulus as a function of  $g$ . It is fixed by the  $1/\tilde{z}^2$ -term of the expansion

$$\tilde{\zeta}(\tilde{z}) = \tilde{z}^2 - \frac{a}{A} - \frac{2}{g^2 A^2} \frac{1}{\tilde{z}^2} - \frac{\frac{1}{2g^2} + 6 \langle \frac{\text{Tr}}{N} A^2 \rangle}{g^2 A^3} \frac{1}{\tilde{z}^4} + \dots \quad (3.18)$$

Comparing the Laurent expansions of  $\tilde{\zeta}(u)$  and  $\tilde{z}(u)$  at the point  $u_\infty = 2iK'$  (the expansion of  $z(u)$  is most easily obtained using the relation (3.18)), we find

$$\frac{g^2}{2} \left( \frac{\pi}{K'} \right)^4 = \left( \frac{2E' - K'}{2K'} + \frac{1 - 2k^2}{3} \right)^2 - \frac{1 - 16k^2 k'^2}{36}. \quad (3.19)$$

The expansion (1.2) of the free energy can be obtained using (2.9) from the  $1/\tilde{z}^4$ -term of the expansion (3.18). The Gaussian limit  $g \rightarrow 0$  corresponds to  $k' = 4g + g^2 + \dots \rightarrow 0$ . Therefore, it is convenient first to express the solution (3.16) in terms of the dual modulus. Using the duality relations

$$H(u, k) = -i\sqrt{K/K'}e^{-\pi u^2/4KK'}H(iu, k'),$$

$$\text{cn}(u, k) = \frac{1}{\text{cn}(iu, k')}, \quad \text{ds}(u, k) = i \text{ds}(iu, k')$$

we write

$$\tilde{z}(u) = i \left[ \frac{H'(iu, k')}{H(iu, k')} - \frac{\text{dn}(iu, k')}{\text{sn}(iu, k')} \right], \quad \tilde{\zeta}(u) = \frac{\text{cn}(iu, k') - 1}{\text{cn}(iu, k') + 1}. \quad (3.20)$$

The large- $N$  limit of the free energy  $\mathcal{F}$  is obtained from the asymptotics (3.18). This is purely technical exercise and we will not do it here. Instead, we will obtain the asymptotical behavior of the free energy when the volume of the graph diverges.

#### 4. Critical behavior

The Taylor expansion of the free energy in  $g$  is convergent up to the critical point  $g_*$ , where the nearest singularity of  $\mathcal{F}(g)$  is located. Since the free energy is an analytic function of  $k^2 = 1 - k'^2$ , the critical point is determined by  $d\beta/dk = 0$ . The function  $\beta(k')$  has a minimum at  $k'^2_* = 0.826114$  where

$$2E(k'_*) = K(k'_*). \quad (4.1)$$

In the vicinity of the critical point ( $k' = k'_* + \delta k'$ )

$$\frac{g^2}{g_*^2} = 1 - \frac{1 - 2(k'_*k'_*)^2}{(k'_*k'_*)^2} \delta k'^2 + \mathcal{O}(\delta k'^4), \quad (4.2)$$

where

$$g_* = \frac{K^2(k'_*)}{\pi^2 \sqrt{6}} = \frac{(K^2)_{2E=K}}{\pi^2 \sqrt{6}}. \quad (4.3)$$

By (3.15) this is exactly the point where the right end of the left cut of  $\zeta(z)$  touches the left end of the right cut. At this point  $u_2^* = u_1^* = 0$ ,  $c^* = a^*$  and  $\Delta^* = 1/2g_*$ . Finally, it is evident that the critical singularity of the free energy is  $\mathcal{F}(g) - \mathcal{F}(g_*) \sim (g_* - g)^{5/2}$ , which implies (1.3).

#### 5. Conclusion

We have found that the coefficients of the free energy (1.2) grow as

$$\mathcal{N}_0(2m) \sim g_*^{-2m} m^{-7/2}.$$

On the other hand, it is known [10] that the number of 3-coordinated planar graph with  $2m$  vertices grows as  $(12\sqrt{3})^n n^{-7/2}$ . Therefore, for a large but finite planar graph, the number of three-colorings grows linearly with the number of vertices  $A = 2m$ , up to a term  $\mathcal{O}(1/A)$ . There are no logarithmic corrections. This means that the number of colorings of a random graph is an extensive quantity and we can define the observable

$$C_3 = \lim_{m \rightarrow \infty} \frac{\#(\text{three-colored planar graphs with } 2m \text{ vertices})}{\#(\text{planar graphs with } 2m \text{ vertices})}$$

representing the number of three-colorings per vertex of an infinite random 3-coordinated planar graph. It follows from our calculation that

$$C_3 = \frac{1/g_*}{\sqrt{12}\sqrt{3}} = 0.984318 \dots \quad (5.1)$$

The fact that the number of three-colorings is slightly less than one is explained by the fact that not all 3-coordinated planar graphs are three-colorable (for example, the graph associated with the tetrahedron is not).

In terms of the  $O(2)$  loop gas, Eq. (5.1) gives the entropy per vertex of the gas fully packed loops on a random 3-coordinated graph, having two different flavors and even length. This is to be compared with the entropy of the gas of fully packed loops with  $n$  flavors and no restriction on the length [11]

$$\frac{1/g_*^{O(n)}}{\sqrt{12}\sqrt{3}} = \frac{2\sqrt{2(2+n)}}{\sqrt{12}\sqrt{3}},$$

which is equal to 1.24081 for  $n = 2$ .

In order to interpret our result in term of the  $O(1)$  loop gas on a 4-coordinated planar graph, we recall that the number of planar 4-coordinated graphs with  $m$  vertices grows as  $12^m m^{-7/2}$  [10]. Therefore the entropy per vertex of the fully packed non-oriented loops on a 4-coordinated random graph is given by

$$\frac{1/g_*^2}{12} = 1.62593 \dots$$

Finally, let us mention that the integral (2.1) is related to Baxter's 6-vertex model on a random lattice, which has been recently solved exactly [12]. In the parameterization

$$a = b = 1, \quad c = -2 \cos \beta$$

of the vertex weights, the integral (2.1) corresponds to the limit  $\beta \rightarrow 0$ , where  $\beta$  is assumed to be purely imaginary. The parameter

$$\Delta = \frac{a_2 + b^2 - c^2}{2ab} = -\cos 2\beta$$

achieves the value  $-1$  from the left, which means that we are approaching the boundary between the regimes III and IV from the side of the regime IV. This regime IV is characterized by a finite correlation length, which is confirmed by our solution. On the other hand, the matrix integral considered in [6] can be identified with the limit  $\beta \rightarrow 0$  with  $\beta > 0$ , which means that the boundary III/IV is achieved from the critical regime III characterized by an infinite correlation length. The fact that the real and imaginary values of the coupling  $g$  are associated with different regimes of the six-vertex model explains why our solution cannot be obtained from the solution found in [6] by analytic continuation  $ig \rightarrow g$ . Even if the perturbative expansions in  $g^2$  are identical, they differ by non-perturbative terms.

### Note added

After this manuscript was to the publisher, the author learned that Bertrand Eynard and Charlotte Kristjansen found independently the exact solution of the problem.

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